ST 1210: Introduction to Probability and Statistics

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Lecture-1b: Basic Probability

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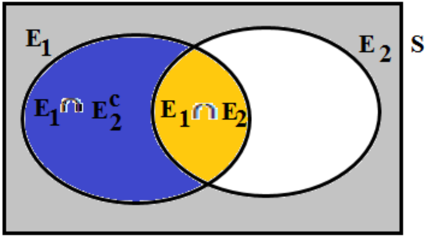
Basic probability

Bayes' theorem

- Consider the events E_1 and E_2 represented by the Venn diagram below



- It follows that
- $E_1 = (E_1 \cap E_2) \cup (E_1 \cap E_2^c)$
- This implies that for an outcome to be in E_1 it must either be in both E_1 and E_2 or in E_1 and not in E_2

Bayes' theorem cont . . .

- Since $E_1 \cap E_2$ and $E_1 \cap E_2^c$ are mutually exclusive then $P(E_1) = P(E_1 \cap E_2) + P(E_1 \cap E_2^c)$
- From $P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$,
- It implies that $P(E_1 \cap E_2) = P(E_2|E_1)P(E_2)$ and $P(E_1 \cap E_2^c) = P(E_1|E_2^c)P(E_2^c)$
- Thus, $P(E_1) = P(E_1|E_2)P(E_2) + P(E_1|E_2^c)P(E_2^c)$
- It follows that, $P(E_1) = P(E_1|E_2)P(E_2) + P(E_1|E_2^c)(1 - P(E_2))$
- Thus, $P(E_2|E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)}$
- It implies that $P(E_2|E_1) = \frac{P(E_2|E_1)P(E_1)}{P(E_1|E_2)P(E_2) + P(E_1|E_2^c)(1 - P(E_2))}$

Bayes' theorem cont ...

Note

- Suppose $B_1, B_2, \ldots, B_n$ are mutually exclusive events such that $\displaystyle\bigcup_{i=1}^{i=n} B_i = S.$
- It implies that exactly one of the events $B_1, B_2, \ldots, B_n$ will occur.
- By writing $E = \displaystyle\bigcup_{i=1}^{i=n} E \cap B_i$ and by considering that $E \cap B_i$ for $i = 1, 2, \ldots, n$ are mutually exclusive events then $P(E) = \displaystyle\sum_{i=1}^{i=n} P(E \cap B_i).$
- It implies that $P(E) = \displaystyle\sum_{i=1}^{i=n} P(E|B_i)P(B_i)$

Bayes' theorem cont ...

- Therefore, $P(B_i|E) = \frac{P(E \cap B_i)}{P(E)} = \frac{P(E|B_i)P(B_i)}{\sum_{i=1}^{i=n} P(E|B_i)P(B_i)}$

Example

Consider two urns. The first contains two white and seven black balls, and the second contains five white and six black balls. We flip a fair coin and then draw a ball from a first urn or the second urn depending on whether the outcome was head or tail. What is the conditional probability that the outcome of the toss was a head given that a white ball was selected?

- Let W be the event that a white ball is drawn and let H be the event that the coin lands with head up, the desired probability $P(H|W)$ can be obtained as follows

$$P(H|W) = \frac{P(H \cap W)}{P(W)} = \frac{P(W|H)P(H)}{P(W)}$$
$$= \frac{P(W|H)P(H)}{P(W|H)P(H) + P(W|H^c)P(H^c)}$$

Bayes' theorem cont ...

- It implies that

$$P(H|W) = \frac{\left(\frac{2}{9} \times \frac{1}{2}\right)}{\left(\frac{2}{9} \times \frac{1}{2}\right) + \left(\frac{5}{11} \times \frac{1}{2}\right)} = \frac{22}{67}$$

Example

In answering a question on a multiple-choice test a student either knows the answer or guesses. Let p be the probability that she knows the answer and $1 - p$ the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $\frac{1}{m}$ where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?

Bayes' theorem cont ...

- Let C and K denote respectively the event that the student answers the question correctly and the event that she actually knows the answer.
- It follows that

$$P(K|C) = \frac{P(K\cap C)}{P(C)} = \frac{P(C|K)P(K)}{P(C|K)P(K)+P(C|K^c)P(K^c)}$$
$$= \frac{p}{p+(\frac{1}{m})(1-p)}$$
$$= \frac{mp}{1+(m-1)p}$$

Exercise

A laboratory blood test is 95% effective in detecting a certain disease when it is, infact, present. However, the test also yields a "false positive" result for 1% of healthy persons tested . (That is, if a healthy person is tested, then, with probability 0.01, the test result will imply he has the disease.) If 0.5% of the population actually has the disease, what is the probability a person has the disease given that his test results is positive?

Fundamental principle of counting

Fundamental principle of counting

Suppose an experiment consists of two parts,if part 1 can result in any of $n-$possible outcomes and if for each outcome of part 1 there are $m-$possible outcomes of part two, then there is a total of $nm-$possible outcomes of the experiment.

Example

One man and one woman are to be selected from the group consisting of 12 women and 8 men. How many different choices are possible?

solution

- Let the choice of a woman be the first part of the experiment and the choice of a man be the second part of the experiment.
- Therefore, there are $12 \cdot 8 = 96$ possible outcomes.

Fundamental principle of counting cont ...

Example

Two people are to be selected from a group that consists of 10 married couples. How many different choices are possible? If each choice is equally likely, what is the probability that the two people selected are married to each other?

Solution

- Since the person selected first is any of 20 people and the next one is any of the remaining 19, it follows from the basic principle that there are $20 \cdot 19 = 130$ possible outcomes.
- Now, for each married couple there are two outcomes that results in the people's selection, the husband could be the first person selected and wife the second or vice-versa
- Thus, there are $2 \cdot 10 = 20$ different outcomes that result in a married couple's selection

Fundamental principle of counting cont ...

- Assuming that all possible outcomes are equally likely, it follows that the probability that the people selected are married to each other is $\frac{20}{380} = \frac{1}{19}$

Generalized basic principle of counting

Suppose an experiment consists of r parts. Suppose there are n_1 possible outcomes of part 1 and then n_2 possible outcomes of part 2 and then n_3 possible outcomes of n_3 and so on, then there is a total of $n_1 \cdot n_2 \cdot n_3 \ldots n_r$ possible outcomes of the experiment.

Example

How many ways can the letters ABC be arranged in a linear order?

- By enumeration we have 6 possible arrangements ABC, ACB, BAC, BCA, CAB and CBA

Fundamental principle of counting cont . . .

- By generalized principle of counting, there are 3 choices for the first character in the ordering,then 2 for the second and 1 for the third
- It implies that there are $3 \cdot 2 \cdot 1 = 6$ possible outcomes.

Note

If we want to determine the number of different arrangements of $n-$objects then by applying the generalized principle of counting we obtain $n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1 = n!$ possible arrangements.

Example

If 4 people are in the room, what is the probability that no two of them celebrate their birthday on the same day of the year?

Solution

- Since each can celebrate at any day of the year then there are

$365 \cdot 365 \cdot 365 \cdot 365 = 365^4 = 17,748,900,625$ possible outcomes

- For two not to have same birthday then there are $365 \cdot 364 \cdot 363 \cdot 362 = 17,458,601,160$ possible outcomes
- Therefore, the probability that no 2 people have the same birthday is $\frac{17,458,601,160}{17,748,900,625} = 0.983644$

Example

At a blood check center, blood can be labelled in one of the four types(A, B, AB or O), one of Rh factors ($+$ or $-$) and one of two genders (F or M). How many different ways can the blood be labelled?

Solution

Blood types:	A, B, AB or O
Rh factors:	Rh^+ or Rh^-
Genders:	F or M

Fundamental principle of counting cont ...

- By using FPC, the blood can be labelled in the following number of ways

Blood type		Rh factors		Gender		Total
A, B, AB or O		Rh^+ or Rh^-		M or F		ways
4	$\times$	2	$\times$	2	$=$	16

- Therefore, there are 16 different ways to label the blood

Permutations and Combinations

- Consider an example of arrangements of letters ABC and of $n-$objects discussed earlier
- Each arrangement in those examples is called a *Permutation*

Permutation

Is the ordered arrangement of a set of distinct objects. An ordered arrangement of $r-$elements of a set is called $r-$permutation

- The number of $r-$permutations of a set with $n-$elements is denoted by $P(n, r)$ or ${}^nP_r$

Theorem

If n is a positive integer and r is an integer such that $r \leq n$, then there are $P(n, r) = n(n-1)(n-2)\ldots(n-r+1)$ with $n-$distinct elements

Permutations and Combinations cont ...

Note

If n and r are integers such that $1 \leq r \leq n$, then $P(n,r) = \frac{n!}{(n-r)!}$

Example

Suppose that a salesman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the salesman use when visiting these cities?

Solution

- The first city is chosen, and the rest is ordered arbitrarily. Hence the orders are, $7! = 5040$

Permutations and Combinations cont ...

Combinations

An $r-$combinations of elements of a set is unordered selection of $r-$elements from the set. Thus an $r-$combination is simply a subset of the set with $r-$elements.

- The number of $r-$combinations of a set with $n-$distinct elements is denoted by $C(n,r)$ or sometimes denoted by ${}^{n}C_{r}$ or $\binom{n}{r}$ and it is given by $C(n,r) = {}^{n}C_{r} = \frac{n!}{(n-r)!r!}$

Activity

Let n and r be non-negative integers with $r \leq n$ show that $C(n,r) = C(n,n-r)$

Example

How many different groups of size 2 can be selected from the items a, b, c ?

Solution

- There are $\binom{3}{2} = \frac{3 \cdot 2}{2 \cdot 1} = 3$ different groups of 2 items.

Example

A committee of 4 people is to be selected from a group of 5 men and 7 women. If the selection is made randomly, what is the probability the committee will consist 2 men and 2 women?

Solution

- Since the selection is done randomly then each of $\binom{12}{4}$ have equal chance to be chosen

Permutations and Combinations cont . . .

- There are $\binom{5}{2}$ possible choices of men and $\binom{7}{2}$ possible choices of women
- From FPC there are $\binom{5}{2}\binom{7}{2}$ possible outcomes containing of 2 men and 2 women
- Therefore, the desired probability is

$$\frac{\binom{5}{2}\binom{7}{2}}{\binom{12}{4}} = \frac{14}{33}$$

Activity

A random sample of size 3 is to be selected from a set of 10 items. What is the probability that a prespecified item will be selected?

Permutations and Combinations cont ...

Note

If we have n elements of which x are alike of one kind, y are alike of another kind, z are alike of another kind then the number of ordered selections or permutations is given by ${}^{n}P_{r} = \dfrac{n!}{x!y!z!}$

Example

How many different arrangements of the word **SYLLABUS** are possible?

- There are $\dfrac{8!}{2!2!} = 10080$ arrangements

Thank you for your Attention.